18.152 PROBLEM SET 4

due April 15th 10:00 am

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

Problem 1. *Suppose that $f, g \in C^\alpha(\Omega)$. Show that*

(1) $[fg]_{\alpha;\Omega} \le [f]_{\alpha;\Omega}[g]_{0;\Omega} + [f]_{0;\Omega}[g]_{\alpha;\Omega}$.
(2) $\|f+g\|_{C^\alpha(\Omega)} \le \|f\|_{C^\alpha(\Omega)} + \|g\|_{C^\alpha(\Omega)}$.

Problem 2. *Let $I = [0,1] \subset \mathbb{R}$.*

(1) *Provide a sequence $\{u_m\}_{m\in\mathbb{N}} \subset C^0(I)$ such that for each $x \in I$ the limit $u(x) = \lim_{m\to+\infty} u_m(x)$ exists and*

$$\|u_m\|_{C^0(I)} \le 1 \quad \text{for all } m \in \mathbb{N}, \qquad\qquad u \notin C^0(I).$$

(2) *Suppose that a sequence $\{u_m\}_{m\in\mathbb{N}} \subset C^0(I)$ satisfies*

$$\lim_{\min\{m,n\}\to+\infty} \|u_m - u_n\|_{C^0(I)} = 0.$$

Show that $u(x) = \lim_{m\to+\infty} u_m(x)$ exists for each $x \in I$ and $u \in C^0(I)$.

(3) *Suppose that given $\alpha \in (0,1]$ a sequence $\{u_m\}_{m\in\mathbb{N}} \subset C^{2,\alpha}(I)$ satisfies*

$$\|u_m\|_{C^\alpha(I)} \le 1,$$

for all $m \in \mathbb{N}$. Show that $\{u_m\}$ has a subsequence $\{u_{m_j}\}$ such that $u(x) = \lim_{m_j\to+\infty} u_{m_j}(x)$ exists for each $x \in I$ and $u \in C^\alpha(I)$.

HINT(3) : Find a subsequence which is convergent at every rational number.

Problem 3 (Schwarz reflection principle)**.** *Suppose that $u \in C^2(\mathbb{R}^n_+) \cap C^0(\overline{\mathbb{R}}^n_+)$ is harmonic in $\mathbb{R}^n_+$ and $u(x',0) = 0$ for all $x' \in \mathbb{R}^{n-1}$. Show that the function*

$$\bar{u}(x', x_n) = \begin{cases} u(x', x_n) & \text{if } x_n \ge 0, \\ -u(x', -x_n) & \text{if } x_n < 0, \end{cases}$$

obtained from u by odd reflection with respect to x_n, is harmonic in $\mathbb{R}^n$.

HINT : Let v be the harmonic function in $B_r(0)$ such that $v = \bar{u}$ on $\partial B_r(0)$. Define $w(x', x_n) = v(x', x_n) + v(x', -x_n)$ and show $w = 0$.

Problem 4. *Show that given* $\alpha \in (0,1)$ *there exists some constant* $C = C(n,\alpha)$ *such that*

$$[D^2 u]_{\alpha;\overline{\mathbb{R}}^n_+} \leq [\Delta u]_{\alpha;\overline{\mathbb{R}}^n_+}$$

holds for every $u \in C^{2,\alpha}(\overline{\mathbb{R}}^n_+)$ *satisfying* $u(x',0) = 0$ *for* $x' \in \mathbb{R}^{n-1}$.

HINT 1 : Use the result in Problem 3.
HINT 2 : Use $u_i(x',0) = u_{jk}(x',0) = 0$ for $i,j,k \in \{1,\cdots,n-1\}$.